

LINEAR MEASURE AND K -QUASICONFORMAL HARMONIC MAPPINGS

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ABSTRACT. In this paper, we investigate the relationships between linear measure and harmonic mappings.

1. PRELIMINARIES AND MAIN RESULTS

For $a \in \mathbb{C}$ and $r > 0$, we let $\mathbb{D}(a, r) = \{z : |z - a| < r\}$ so that $\mathbb{D}_r := \mathbb{D}(0, r)$ and thus, $\mathbb{D} := \mathbb{D}_1$ denotes the open unit disk in the complex plane \mathbb{C} . Let $\mathbb{T} = \partial\mathbb{D}$ be the boundary of \mathbb{D} . For a real 2×2 matrix A , we use the matrix norm $\|A\| = \sup\{|Az| : |z| = 1\}$ and the matrix function $\lambda(A) = \inf\{|Az| : |z| = 1\}$. For $z = x + iy \in \mathbb{C}$, the formal derivative of the complex-valued functions $f = u + iv$ is given by

$$D_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

so that

$$\|D_f\| = |f_z| + |f_{\bar{z}}| \quad \text{and} \quad \lambda(D_f) = ||f_z| - |f_{\bar{z}}||,$$

where $f_z = (1/2)(f_x - if_y)$ and $f_{\bar{z}} = (1/2)(f_x + if_y)$ are partial derivatives.

Let Ω be a domain in \mathbb{C} , with non-empty boundary. A sense-preserving homeomorphism f from a domain Ω onto Ω' , contained in the Sobolev class $W_{loc}^{1,2}(\Omega)$, is said to be a K -quasiconformal mapping if, for $z \in \Omega$,

$$\|D_f(z)\|^2 \leq K \det D_f(z), \quad \text{i.e., } \|D_f(z)\| \leq K \lambda(D_f(z)),$$

where $K \geq 1$ and $\det D_f$ denotes the determinant of D_f (cf. [9, 12, 18]). We note that $\det D_f = |f_z|^2 - |f_{\bar{z}}|^2$, the Jacobian of f , is usually denoted by J_f .

A complex-valued function f defined in a simply connected subdomain G of \mathbb{C} is called a *harmonic mapping* in G if and only if both the real and the imaginary parts of f are real harmonic in G . It is well known that every harmonic mapping f in G admits a decomposition $f = h + \bar{g}$, where h and g are analytic in G . Throughout we use this representation. Without loss of generality, we assume $0 \in G$. If we choose the additive constant such that $g(0) = 0$, then the decomposition is unique. Because $J_f = |h'|^2 - |g'|^2$, it follows that f is locally univalent and sense-preserving in G if and only if $|g'(z)| < |h'(z)|$ in G ; or equivalently if $h'(z) \neq 0$ and the dilatation $\omega = g'/h'$ has the property that $|\omega(z)| < 1$ in G (see [6] and also [13]).

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Let $\gamma : \varphi(t)$, $t \in [\alpha, \beta]$, be a curve in \mathbb{C} . Its length $\ell(\gamma)$ is defined by

$$(1.1) \quad \ell(\gamma) = \sup \sum_{k=1}^n |\varphi(t_k) - \varphi(t_{k-1})|,$$

where the supremum is taken over all partitions $\alpha = t_0 < t_1 < \dots < t_n = \beta$ and all $n \in \{1, 2, \dots\}$. We call γ *rectifiable* if $\ell(\gamma) < +\infty$. It is clear from (1.1) that $\text{diam} \gamma \leq \ell(\gamma)$. In the case of a closed curve $\gamma : \psi(\zeta)$, $\zeta \in \mathbb{T}$, with piecewise continuously differentiable ψ , we can write

$$\ell(\gamma) = \int_{\mathbb{T}} |\psi'(\zeta)| |d\zeta|,$$

where $\mathbb{T} = \partial\mathbb{D}$. If $\gamma_1, \gamma_2, \dots$ are disjoint curves, then we define

$$\ell(\cup_k^\infty \gamma_k) = \sum_{k=1}^\infty \ell(\gamma_k).$$

In particular, $\ell(\emptyset) = 0$ (see [16, p. 3]).

For $p \in (0, \infty]$, the *generalized Hardy space* $H_g^p(\mathbb{D})$ consists of all those functions $f : \mathbb{D} \rightarrow \mathbb{C}$ such that f is measurable, $M_p(r, f)$ exists for all $r \in (0, 1)$ and $\|f\|_p < \infty$, where

$$\|f\|_p = \begin{cases} \sup_{0 < r < 1} M_p(r, f) & \text{if } p \in (0, \infty) \\ \sup_{z \in \mathbb{D}} |f(z)| & \text{if } p = \infty \end{cases}, \quad \text{and} \quad M_p^p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta.$$

Proposition 1. *Let f be a K -quasiconformal harmonic mapping of \mathbb{D} onto an inner domain of Jordan curve γ . Then $\ell(\gamma) < +\infty$ if and only if $\|D_f\| \in H_g^1(\mathbb{D})$.*

Proof. Assume that $f = h + \bar{g}$, where h and g are analytic in \mathbb{D} . We first prove the necessity. Let $\ell(\gamma) < +\infty$. Then, by Lemma B (in Section 2), we see that

$$F_n(z) = \sum_{k=1}^n |f(ze^{2\pi ki/n}) - f(ze^{2\pi(k-1)i/n})|$$

is subharmonic in \mathbb{D} and continuous in $\overline{\mathbb{D}}$. It follows from the maximum principle and (1.1) that $F_n(z) \leq \ell(\gamma)$ for $z \in \mathbb{D}$, and thus, for $r \in [0, 1)$, we have

$$\begin{aligned} \frac{r}{K} \int_0^{2\pi} \|D_f(re^{it})\| dt &\leq r \int_0^{2\pi} \left| h'(re^{it}) - e^{-2it} \overline{g'(re^{it})} \right| dt \\ &= \int_0^{2\pi} |df(re^{it})| = \lim_{n \rightarrow \infty} F_n(r) \\ &\leq \ell(\gamma), \end{aligned}$$

which implies that $\|D_f\| \in H_g^1(\mathbb{D})$.

Next, we prove the sufficiency part. For $r \in [0, 1)$, let $\gamma_r = \{f(re^{i\theta}) : \theta \in [0, 2\pi)\}$. Since $|zh'(z) - \overline{zg'(z)}|$ is subharmonic in \mathbb{D} , we see that

$$\ell(\gamma_r) = r \int_0^{2\pi} \left| h'(re^{it}) - e^{-2it} \overline{g'(re^{it})} \right| dt$$

is an increasing function of r on $[0, 1)$. By calculations, we get

$$\ell(\gamma_r) \leq \int_0^{2\pi} \|D_f(re^{it})\| dt \leq \sup_{0 < r < 1} \int_0^{2\pi} \|D_f(re^{it})\| dt < +\infty,$$

which, together with the monotonicity, yields that $\lim_{r \rightarrow 1^-} \ell(\gamma_r)$ does exist and thus,

$$\ell(\gamma) \leq \lim_{r \rightarrow 1^-} \ell(\gamma_r) < +\infty,$$

as desired. \square

In [11], Lavrentiev proved that if f maps \mathbb{D} conformally onto the inner domain of Jordan curve γ of finite length, then, for any $E \subset \gamma$, $\ell(E) > 0$ implies that $\ell(f(E)) > 0$. For univalent harmonic mappings, we obtain the following result.

Theorem 1. *Let f be a sense-preserving and univalent harmonic mapping from $\overline{\mathbb{D}}$ onto a domain Ω and $\partial\Omega$ is a rectifiable Jordan curve. Furthermore, let $E \subset \mathbb{T}$ is measurable with $\ell(E) > 0$. Then, we have*

$$\ell(f(E)) \geq \frac{\ell(\partial\Omega)\ell(E)}{2\pi - \ell(E)} \left[\frac{(|f_z(0)| - |f_{\bar{z}}(0)|)(2\pi - \ell(E))}{\ell(\partial\Omega)} \right]^{\frac{2\pi}{\ell(E)}}.$$

This estimate is sharp as $\ell(E) \rightarrow 2\pi^-$ and the extreme univalent harmonic mapping is

$$f(z) = \frac{M}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} e^{i\varphi(t)} dt$$

satisfying $|f_z(0)| - |f_{\bar{z}}(0)| = M$, where M is a positive constant and $\varphi(t)$ is a continuously increasing function in $[0, 2\pi]$ with $\varphi(2\pi) - \varphi(0) = 2\pi$.

Corollary 1. *Under the hypotheses of Theorem 1, we have $\ell(f(E)) > 0$.*

Let γ be a rectifiable Jordan curve. The shorter arc between z and w in γ will be denoted by $\gamma[z, w]$. We say that γ is a M -Lavrentiev curve if there is a constant $M > 1$ such that $\ell(\gamma[z, w]) \leq M|z - w|$ for each $z, w \in \gamma$. The inner domain of a M -Lavrentiev curve is called a M -Lavrentiev domain (cf. [7, 9, 16, 19]).

The following result is considered to be a Schwarz-type lemma of subharmonic functions.

Theorem A. ([1, Theorem 2]) *Let ϕ be subharmonic in \mathbb{D} . If, for all $r \in [0, 1)$,*

$$A(r) = \sup_{\theta \in [0, 2\pi]} \int_0^r \phi(\rho e^{i\theta}) d\rho \leq 1,$$

then $A(r) \leq r$.

Analogy to Theorem A, applying some part of proof technique of [19, Lemma 1], we obtain a Schwarz type estimate on the length function of K -quasiconformal harmonic mappings.

Theorem 2. Suppose that f is a K -quasiconformal harmonic mapping of \mathbb{D} onto a M -Lavrentiev domain Ω . Then, for $r \in (0, 2]$, $\rho \in (0, r]$ and any fixed $\zeta_0 \in \partial\mathbb{D}$,

$$\int_0^r \ell(f(\Gamma_\rho)) d\rho \leq \sqrt{\frac{K\pi A(\Omega)}{3}} \frac{r^{\frac{3}{2}}}{e^{\frac{\alpha}{2}(\frac{1}{r}-\frac{1}{2})}} \leq \sqrt{\frac{K\pi A(\Omega)}{3}} r^{\frac{3}{2}},$$

where $\alpha = 4/[K(1+M)^2]$, $A(\Omega)$ is the area of Ω and Γ_ρ is the arc of the circle $\partial\mathbb{D}(\zeta_0, \rho)$ which lies in $\overline{\mathbb{D}}$.

We say that a simply connected domain $G \subset \mathbb{C}$ is M -linearly connected if, for any two points $z_1, z_2 \in G$, there is a curve $\gamma \subset G$ and a constant $M \geq 1$ such that

$$(1.2) \quad \text{diam} \gamma \leq M|z_1 - z_2|.$$

Zinsmeister [20] obtained an analytic characterization of M -Lavrentiev domains (see also [16, Chapter 7]). The following result is an analogous result to the analytic characterization of M -Lavrentiev domains.

Theorem 3. Let f be a K -quasiconformal harmonic mapping from \mathbb{D} onto the inner domain G of a rectifiable Jordan curve. If G is a M_1 -Lavrentiev domain and for each $\zeta \in \partial\mathbb{D}$,

$$(1.3) \quad \|D_f(\rho\zeta)\| \leq M'_1 \|D_f(r\zeta)\| \left(\frac{1-\rho}{1-r} \right)^{\delta-1} \quad (0 \leq r \leq \rho < 1),$$

then G is M_2 -linearly connected and, for all $z \in \mathbb{D}$, there is a constant M' such that

$$(1.4) \quad \frac{1}{\ell(I(z))} \int_{I(z)} \|D_f(\zeta)\| |d\zeta| \leq M' \|D_f(z)\|,$$

where $I(z) = \{\zeta \in \mathbb{T} : |\arg \zeta - \arg z| \leq \pi(1-|z|)\}$ and $\delta \in (0, 1)$, M'_1 , M_1 , M_2 are constants.

For any fixed $\theta \in [0, 2\pi]$, the radial length of the curve $C_\theta(r) = \{w = f(\rho e^{i\theta}) : 0 \leq \rho \leq r\}$ with counting multiplicity is defined by

$$\ell_f^*(\theta, r) = \int_0^r |df(\rho e^{i\theta})| = \int_0^r |f_z(\rho e^{i\theta}) + e^{-2i\theta} f_{\bar{z}}(\rho e^{i\theta})| d\rho,$$

where $r \in [0, 1)$ and f is a harmonic mapping defined in \mathbb{D} . In particular, let

$$\ell_f^*(\theta, 1) = \sup_{0 < r < 1} \ell_f^*(\theta, r).$$

Proposition 2. Suppose that f is a bounded harmonic mapping in \mathbb{D} and $r_0 \in (0, 1)$. For $\zeta \in \mathbb{D}$, let $F(\zeta) = f(r_0\zeta)$. Then, for $\rho \in [0, 1)$ and $\theta \in [0, 2\pi]$,

$$\ell_F^*(\theta, r) = \int_0^r r_0 |f_z(\rho r_0 e^{i\theta}) + e^{-2i\theta} f_{\bar{z}}(\rho r_0 e^{i\theta})| d\rho \leq Mr,$$

where $r \in (0, 1)$ and $M = \frac{2r_0}{\pi} \sup_{z \in \mathbb{D}} |f(z)| \log((1+r_0)/(1-r_0))$.

Proof. For $\zeta \in \mathbb{D}$, let $F(\zeta) = f(r_0\zeta)$. By [5, Theorem 3], we get

$$\begin{aligned}\ell_F^*(\theta, r) &= \int_0^r r_0 |f_z(\rho r_0 e^{i\theta}) + e^{-2i\theta} f_{\bar{z}}(\rho r_0 e^{i\theta})| d\rho \\ &\leq \int_0^r r_0 \|D_f(r_0 \rho e^{i\theta})\| d\rho \\ &\leq \frac{4r_0 \sup_{z \in \mathbb{D}} |f(z)|}{\pi} \int_0^r \frac{d\rho}{1 - r_0^2 \rho^2} \\ &= \frac{2r_0 \sup_{z \in \mathbb{D}} |f(z)|}{\pi} \log \left(\frac{1 + r_0 r}{1 - r_0 r} \right) \\ &\leq \frac{2r_0 \sup_{z \in \mathbb{D}} |f(z)|}{\pi} \log \left(\frac{1 + r_0}{1 - r_0} \right) = M.\end{aligned}$$

By the subharmonicity of $\|D_f(r_0 \rho e^{i\theta})\|$ and Theorem A, we see that, for $\theta \in [0, 2\pi]$, $\ell_F^*(\theta, r) \leq Mr$, where $r \in (0, 1)$. \square

In [3], the authors obtained the coefficient estimates of a class of K -quasiconformal harmonic mapping with a finite perimeter length. In the following, we will investigate the coefficient estimates on a class of K -quasiconformal harmonic mapping with the finite radial length.

Theorem 4. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \bar{b}_n \bar{z}^n$ be a K -quasiconformal harmonic mapping on \mathbb{D} . If, for all $\theta \in [0, 2\pi]$, $\ell_f^*(\theta, 1) \leq M$ for some positive constant M , then*

$$(1.5) \quad |a_n| + |b_n| \leq KM \quad \text{for } n \geq 1,$$

In particular, if $K = 1$, then the estimate (1.5) is sharp and the extreme function is $f(z) = Mz$.

Let $d_{\Omega}(z)$ be the Euclidean distance from z to the boundary $\partial\Omega$ of the domain Ω . In the following, we investigate the behaviour on the ratio of the radial length and the perimeter length on K -quasiconformal harmonic mappings.

Theorem 5. *Let f be a K -quasiconformal harmonic mapping of \mathbb{D} onto a bounded domain D . Then, for $r \in (0, 1)$ and all $\theta \in [0, 2\pi]$,*

$$\frac{\ell_f^*(\theta, r)}{\ell(r)} \leq \frac{32r(1+r)K^3 \sup_{z \in \mathbb{D}} |f(z)|}{\int_0^{2\pi} d_D(f(re^{it})) dt}$$

and

$$\lim_{r \rightarrow 0^+} \left\{ \frac{\sup_{\theta \in [0, 2\pi]} \ell_f^*(\theta, r)}{\ell(r)} \right\} = 0.$$

The proofs of Theorems 1, 2, 3, 4 and 5 will be presented in Section 2.

2. THE PROOFS OF THE MAIN RESULTS

The following lemmas are well-known.

Lemma B. (see [8], [9, Proposition 1.8] and [15]) *If $f = h + \bar{g}$ is a K -quasiconformal harmonic mapping of \mathbb{D} onto a Jordan domain with a rectifiable boundary, then h and g have absolutely continuous extension to \mathbb{T} .*

Lemma C. (cf. [17]) *Let (Ω, A, μ) be a measure space such that $\mu(\Omega) = 1$. If g is a real-valued function that is μ -integrable, and if φ is a convex function on the real line, then*

$$\varphi \left(\int_{\Omega} g d\mu \right) \leq \int_{\Omega} \varphi \circ g d\mu.$$

Lemma D. (cf. [2]) *Among all rectifiable Jordan curves of a given length, the circle has the maximum interior area.*

Proof of Theorem 1. Let $f = h + \bar{g}$ be a sense-preserving and univalent harmonic mapping in $\overline{\mathbb{D}}$, where h and g are analytic in $\overline{\mathbb{D}}$. Then, $h'(z) \neq 0$ for $z \in \mathbb{D}$ and the dilatation ω defined by $\omega = g'/h'$ is analytic and $|\omega(z)| < 1$ in \mathbb{D} . Since $\log |h'|$ is harmonic in \mathbb{D} and $\log(1 - |\omega|)$ is subharmonic in \mathbb{D} , we see that, for $r \in (0, 1)$,

$$\log |h'(0)| = \frac{1}{2\pi} \int_E \log |h'(rz)| |dz| + \frac{1}{2\pi} \int_{\mathbb{T} \setminus E} \log |h'(rz)| |dz|$$

and

$$\log(1 - |\omega(0)|) \leq \frac{1}{2\pi} \int_E \log(1 - |\omega(rz)|) |dz| + \frac{1}{2\pi} \int_{\mathbb{T} \setminus E} \log(1 - |\omega(rz)|) |dz|,$$

which, together with Lemma C (Jensen's inequality), imply that

$$\begin{aligned} & 2\pi \log [|h'(0)|(1 - |\omega(0)|)] \\ & \leq \int_E \log [|h'(rz)|(1 - |\omega(rz)|)] |dz| + \int_{\mathbb{T} \setminus E} \log [|h'(rz)|(1 - |\omega(rz)|)] |dz| \\ & \leq \ell(E) \log \left[\frac{1}{\ell(E)} \int_E (|h'(rz)| - |g'(rz)|) |dz| \right] \\ & \quad + (2\pi - \ell(E)) \log \left[\frac{1}{2\pi - \ell(E)} \int_{\mathbb{T} \setminus E} (|h'(rz)| - |g'(rz)|) |dz| \right] \\ & \leq \ell(E) \log \left[\frac{1}{r\ell(E)} \int_E r |h'(rz) - \overline{g'(rz)} \bar{z}^2| |dz| \right] \\ & \quad + (2\pi - \ell(E)) \log \left[\frac{1}{r[2\pi - \ell(E)]} \int_{\mathbb{T} \setminus E} r |h'(rz) - \overline{g'(rz)} \bar{z}^2| |dz| \right]. \end{aligned}$$

By letting $r \rightarrow 1^-$ and Lemma B, we have

$$\begin{aligned} 2\pi \log [|h'(0)|(1 - |\omega(0)|)] &\leq \ell(E) \log \frac{\ell(f(E))}{\ell(E)} + (2\pi - \ell(E)) \log \frac{\ell(f(\mathbb{T} \setminus E))}{(2\pi - \ell(E))} \\ &\leq \ell(E) \log \ell(f(E)) - \ell(E) \log \ell(E) \\ &\quad - (2\pi - \ell(E)) \log(2\pi - \ell(E)) + (2\pi - \ell(E)) \log L, \end{aligned}$$

which implies that

$$(2.1) \quad \ell(f(E)) \geq \frac{L\ell(E)}{2\pi - \ell(E)} \left[\frac{(|h'(0)| - |g'(0)|)(2\pi - \ell(E))}{L} \right]^{\frac{2\pi}{\ell(E)}},$$

where $\ell(\partial\Omega) = L$. Now we prove the sharpness part. By calculation, we have

$$\lim_{\ell(E) \rightarrow 2\pi^-} \frac{L\ell(E)}{2\pi - \ell(E)} \left[\frac{(|h'(0)| - |g'(0)|)(2\pi - \ell(E))}{L} \right]^{\frac{2\pi}{\ell(E)}} = 2\pi(|h'(0)| - |g'(0)|).$$

Let

$$(2.2) \quad f(z) = \frac{M}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} e^{i\varphi(t)} dt$$

satisfying $|f_z(0)| - |f_{\bar{z}}(0)| = M$, where M is a positive constant and $\varphi(t)$ is a continuously increasing function in $[0, 2\pi]$ with $\varphi(2\pi) - \varphi(0) = 2\pi$. If $\ell(E) \rightarrow 2\pi^-$, then the sense-preserving and univalent harmonic mapping f of (2.2) shows that the estimate of (2.1) is sharp. \square

Proof of Theorem 2. Let $f = h + \bar{g}$ satisfy the assumption, where h and g are analytic in \mathbb{D} . By Lemma B, we know that h and g can be absolutely continuous extension to \mathbb{T} . For $r \in (0, 2]$ and $\rho \in (0, r]$, let $\Delta_\rho = \{z : |z - \zeta_0| \leq \rho \text{ and } |z| \leq 1\}$. Let Γ_ρ denote the arc of the circle $\partial\mathbb{D}(\zeta_0, \rho)$ which lies in $\overline{\mathbb{D}}$. Then we have

$$\begin{aligned} \ell^2(f(\Gamma_\rho)) &= \left(\int_{\Gamma_\rho} |f_z(\zeta_0 + \rho e^{it}) - e^{-2it} f_{\bar{z}}(\zeta_0 + \rho e^{it})| \rho dt \right)^2 \\ &\leq \left(\int_{\Gamma_\rho} \rho dt \right) \left(\int_{\Gamma_\rho} |f_z(\zeta_0 + \rho e^{it}) - e^{-2it} f_{\bar{z}}(\zeta_0 + \rho e^{it})|^2 \rho dt \right) \\ &= \ell(\Gamma_\rho) \rho \int_{\Gamma_\rho} |f_z(\zeta_0 + \rho e^{it}) - e^{-2it} f_{\bar{z}}(\zeta_0 + \rho e^{it})|^2 \rho dt \\ &\leq 2\rho^2 \arccos \frac{\rho}{2} \int_{\Gamma_\rho} \|D_f(\zeta_0 + \rho e^{it})\|^2 \rho dt \\ &\leq K\pi\rho^2 \int_{\Gamma_\rho} J_f(\zeta_0 + \rho e^{it}) \rho dt, \end{aligned}$$

which implies that

$$\begin{aligned} (2.3) \quad P(r) = \int_0^r \frac{\ell^2(f(\Gamma_\rho))}{\rho^2} d\rho &\leq K\pi \int_0^r \int_{\Gamma_\rho} J_f(\zeta_0 + \rho e^{it}) \rho dt d\rho \\ &\leq K\pi A_f(r), \end{aligned}$$

where $z = \zeta_0 + \rho e^{it}$ and $A_f(r)$ denotes the area of $f(\Delta_r)$. Since the boundary of Ω is a M -Lavrentiev curve, we see that

$$\ell(f(\partial\Delta_r)) \leq \ell(f(\Gamma_r)) + M\ell(f(\Gamma_r)) = (1+M)\ell(f(\Gamma_r))$$

and thus, by Lemma D, we have

$$(2.4) \quad A_f(r) \leq \frac{(1+M)^2}{4\pi} \ell^2(f(\Gamma_r)).$$

By (2.3) and (2.4), we obtain

$$P(r) \leq \frac{K(1+M)^2}{4} \ell^2(f(\Gamma_r)).$$

By calculations, for $r \in (0, 2]$, we have

$$\frac{4}{K(1+M)^2} P(r) \leq \ell^2(f(\Gamma_r)) = r^2 P'(r),$$

which gives that

$$(2.5) \quad \frac{\alpha}{r^2} \leq \frac{P'(r)}{P(r)},$$

where $\alpha = 4/[K(1+M)^2]$. By (2.5), we get

$$\int_r^2 \frac{\alpha}{\rho^2} d\rho \leq \int_r^2 \frac{P'(\rho)}{P(\rho)} d\rho,$$

which by integration, together with (2.3), yield that

$$(2.6) \quad P(r) \leq \frac{P(2)}{e^{\alpha(\frac{1}{r}-\frac{1}{2})}} \leq \frac{K\pi A(\Omega)}{e^{\alpha(\frac{1}{r}-\frac{1}{2})}},$$

where $A(\Omega)$ is the area of Ω .

By Cauchy-Schwarz's inequality and (2.6), we have

$$\left(\int_0^r \ell(f(\Gamma_\rho)) d\rho \right)^2 \leq P(r) \int_0^r \rho^2 d\rho \leq \frac{K\pi A(\Omega) r^3}{3e^{\alpha(\frac{1}{r}-\frac{1}{2})}},$$

which gives that

$$\int_0^r \ell(f(\Gamma_\rho)) d\rho \leq \sqrt{\frac{K\pi A(\Omega)}{3}} \frac{r^{\frac{3}{2}}}{e^{\frac{\alpha}{2}(\frac{1}{r}-\frac{1}{2})}} \leq \sqrt{\frac{K\pi A(\Omega)}{3}} r^{\frac{3}{2}}.$$

The proof of this theorem is complete. \square

A Jordan curve J is said to be a M -quasicircle if, for any $z_1, z_2 \in J$, there is a constant $M \geq 1$ such that

$$(2.7) \quad \text{diam} J(z_1, z_2) \leq M|z_1 - z_2|.$$

The inner domain G of a quasicircle J is called a M -quasidisk. We say that the curve $\gamma \subset \mathbb{C}$ is *Ahlfors-regular* if there is a positive constant M such that

$$\ell(\gamma \cap \mathbb{D}(w, r)) \leq Mr,$$

where $r \in (0, \infty)$ and $w \in \mathbb{C}$ (cf. [16]).

Lemma E. ([16, Proposition 7.7]) *A domain is a M_3 -Lavrentiev domain if and only if it is an M_4 -Ahlfors-regular quasidisk, where M_3 and M_4 are positive constants.*

Proof of Theorem 3. Assume that $f = h + \bar{g}$ is a K -quasiconformal harmonic mapping from \mathbb{D} onto the inner domain G of a rectifiable Jordan curve, where h and g are analytic in \mathbb{D} . Let $w_1, w_2 \in G$ and let $[w_1, w_2]$ denote the line segment with endpoints w_1 and w_2 . If $[w_1, w_2] \subset G$, then (1.2) holds. Without loss of generality, we assume that $[w_1, w_2] \not\subset G$. Let $f(\xi_k)$ be the boundary point on $[w_1, w_2]$ nearest to w_k , where $k = 1, 2$ and $\xi_k \in \mathbb{T}$. By (2.7), we see that one of the arcs $\mathbb{T}_{[\xi_1, \xi_2]}$ of \mathbb{T} from ξ_1 to ξ_2 satisfies

$$\text{diam} f(\mathbb{T}_{[\xi_1, \xi_2]}) \leq M_1 |f(\xi_1) - f(\xi_2)|.$$

If r is close enough to 1, then $f(r\mathbb{T}_{[\xi_1, \xi_2]})$ is a curve in G of

$$\text{diam} f(r\mathbb{T}_{[\xi_1, \xi_2]}) \leq 2M_1 |f(\xi_1) - f(\xi_2)| < 2M_1 |w_1 - w_2|$$

which can be connected within G by curves γ_k to w_k satisfying

$$\ell(\gamma_k) < |w_1 - w_2|.$$

Then $\gamma = \gamma_1 \cup f(r\mathbb{T}_{[\xi_1, \xi_2]}) \cup \gamma_2$ is curve in G from w_1 to w_2 satisfying

$$\text{diam} \gamma \leq (2M_1 + 2) |w_1 - w_2|,$$

which implies that G is a $(2M_1 + 2)$ -linearly connected domain.

Now we prove (1.4). Let $I(z) = \{\zeta \in \mathbb{T} : |\arg \zeta - \arg z| \leq \pi(1 - |z|)\}$. By (1.3) and [4, Theorems 1 and 2], we see that there is a constant M^* such that

$$(2.8) \quad \text{diam} f(I(z)) \leq M^* d_G(z).$$

Applying the inequality (2.3) in [4], we get

$$(2.9) \quad d_G(z) \leq \frac{2K}{1+K} \|D_f(z)\| (1 - |z|^2).$$

It follows from (2.8) and (2.9) that

$$(2.10) \quad \text{diam} f(I(z)) \leq \frac{2KM^*}{1+K} \|D_f(z)\| (1 - |z|^2) = r_z,$$

which implies that $f(I(z))$ lies in \mathbb{D}_{r_z} . By (2.10) and Lemma E, we conclude that there is a constant M' such that

$$\begin{aligned} \frac{1}{K} \int_{I(z)} \|D_f(\zeta)\| |d\zeta| &\leq \int_{I(z)} \left| h'(\zeta) - \bar{\zeta}^2 \overline{g'(\zeta)} \right| |d\zeta| = \ell(f(I(z))) \\ &\leq M' \|D_f(z)\| (1 - |z|^2) \\ &\leq M' \ell(I(z)) \|D_f(z)\|. \end{aligned}$$

The proof of the theorem is complete. \square

Theorem F. ([10, Proposition 3.1] and [10, Theorem 3.2]) *Let f be a K -quasiconformal harmonic mapping from \mathbb{D} onto itself. Then for all $z \in \mathbb{D}$, we have*

$$\frac{1+K}{2K} \left(\frac{1 - |f(z)|^2}{1 - |z|^2} \right) \leq |f_z(z)| \leq \frac{K+1}{2} \left(\frac{1 - |f(z)|^2}{1 - |z|^2} \right).$$

Proof of Theorem 4. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \bar{b}_n \bar{z}^n$ be a K -quasiconformal harmonic mapping on \mathbb{D} . Then, by Cauchy's integral formula, for $\rho \in (0, 1)$ and $n \geq 1$, we get

$$na_n = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{f_z(z)}{z^n} dz \quad \text{and} \quad nb_n = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{\overline{f_{\bar{z}}(z)}}{z^n} dz,$$

which imply that

$$\begin{aligned} (2.11) \quad n(|a_n| + |b_n|) &= \frac{1}{2\pi} \left| \int_{|z|=\rho} \frac{f_z(z)}{z^n} dz \right| + \frac{1}{2\pi} \left| \int_{|z|=\rho} \frac{\overline{f_{\bar{z}}(z)}}{z^n} dz \right| \\ &\leq \frac{1}{2\pi\rho^{n-1}} \int_0^{2\pi} \|D_f(\rho e^{i\theta})\| d\theta. \end{aligned}$$

By calculations, for $\theta \in [0, 2\pi]$, we obtain

$$\begin{aligned} \ell_f^*(\theta, r) &= \int_0^r |f_z(\rho e^{i\theta}) + e^{-2i\theta} \overline{f_{\bar{z}}(\rho e^{i\theta})}| d\rho \\ &\geq \int_0^r \lambda(D_f)(\rho e^{i\theta}) d\rho \\ &\geq \frac{1}{K} \int_0^r \|D_f(\rho e^{i\theta})\| d\rho, \end{aligned}$$

which gives

$$(2.12) \quad \int_0^r \|D_f(\rho e^{i\theta})\| d\rho \leq K\ell_f^*(\theta, r) \leq KM.$$

By (2.12), the subharmonicity of $D_f(\rho e^{i\theta})$ and Theorem A, we have

$$(2.13) \quad \int_0^r \|D_f(\rho e^{i\theta})\| d\rho \leq KM r.$$

By (2.11) and (2.13), we get

$$\begin{aligned} 2\pi n(|a_n| + |b_n|) \int_0^r \rho^{n-1} d\rho &= \int_0^r \left(\int_0^{2\pi} \|D_f(\rho e^{i\theta})\| d\theta \right) d\rho \\ &= \int_0^{2\pi} \left(\int_0^r \|D_f(\rho e^{i\theta})\| d\rho \right) d\theta \\ &\leq 2\pi KM r, \end{aligned}$$

which yields that

$$|a_n| + |b_n| \leq \frac{KM}{r^{n-1}} \quad \text{for } n \geq 1.$$

Since this is true for each $r < 1$, the desired bound follows by letting $r \rightarrow 1^-$. \square

Proof of Theorem 5. Let $f = h + \bar{g}$ be a K -quasiconformal harmonic mapping of \mathbb{D} onto a bounded domain D , where h and g are analytic in \mathbb{D} . By [14, Proposition 13] and Theorem F, for $r \in (0, 1)$ and all $\theta \in [0, 2\pi]$, we obtain

$$\begin{aligned}
 (2.14) \quad \ell_f^*(\theta, r) &= \int_0^r |df(\rho e^{i\theta})| = \int_0^r \left| h'(\rho e^{i\theta}) + e^{-2i\theta} \overline{g'(\rho e^{i\theta})} \right| d\rho \\
 &\leq \int_0^r \|D_f(\rho e^{i\theta})\| d\rho \\
 &\leq 16K \int_0^r \frac{d_D(f(\rho e^{i\theta}))}{1 - \rho^2} d\rho \\
 &\leq 16K \sup_{z \in \mathbb{D}} |f(z)| \int_0^r \frac{1}{1 - \rho^2} d\rho \\
 &= 8K \sup_{z \in \mathbb{D}} |f(z)| \log \frac{1+r}{1-r}
 \end{aligned}$$

and

$$\begin{aligned}
 (2.15) \quad \ell(r) &= \int_0^{2\pi} r \left| h'(re^{it}) - e^{-2it} \overline{g'(re^{it})} \right| dt \\
 &\geq r \int_0^{2\pi} \lambda(D_f)(re^{it}) dt \\
 &\geq \frac{r}{K} \int_0^{2\pi} \|D_f(re^{it})\| dt \\
 &\geq \frac{r(1+K)}{2K^2(1-r^2)} \int_0^{2\pi} d_D(f(re^{it})) dt
 \end{aligned}$$

where the last inequality is a consequence of [4, Inequality (2.3)]. Equations (2.14) and (2.15) imply that

$$\frac{\ell_f^*(\theta, r)}{\ell(r)} \leq 16K^3 \sup_{z \in \mathbb{D}} |f(z)| \frac{(1-r^2) \log \frac{1+r}{1-r}}{\int_0^{2\pi} d_D(f(re^{it})) dt} \leq \frac{32r(1+r)K^3 \sup_{z \in \mathbb{D}} |f(z)|}{\int_0^{2\pi} d_D(f(re^{it})) dt}$$

and, for all $\theta \in [0, 2\pi]$,

$$\lim_{r \rightarrow 0^+} \frac{\ell_f^*(\theta, r)}{\ell(r)} = 0.$$

The proof of this theorem is complete. \square

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